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Estimator Type and Population Size for Estimating the Weibull Modulus in Ceramics

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Abstract

Analysis of fracture data in ceramics can be made by applying probabilistic theories. In order the fracture probability to be expressed an estimator must be used. For economic reasons it is also necessary, the population size for estimating the shape ceramic parameter, to be found. In the present work a number of estimators and sets of different sample populations of kaolin ceramics are tested against the stability of the weibull shape parameter. The 2-parameter and 3-parameter Weibull functions and the Neville function were compared for best fitting of results. It seems that modulus remains almost constant in the range of the specimen numbers tested and a minimum number of about 30 specimens is acceptable for reliability predictions at least for the examined type of ceramic.

1 Introduction

The statistical function most commonly used in analysis of strength data of ceramics is the cumulative distribution function proposed by Weibull.[1] using the theorem of the product of independent event probabilities (the probability of an event comprising a number of independent events is equal to the product of the probabilities of the individual events considered separately).

Generally, the Weibull function, is a three parameter equation and it has the following form:

$$P_f = 1 - e^{-\int_x \left[\frac{x-x_u}{x_u}\right]^m dy} \quad \text{for } x \succ x_u$$
$$P_f = 0 \quad \text{for } x \leq x_m$$

where:

P_f is the distribution function of a variable x (probability of failure),

x is the measured variable, x_u is a value of x where the function is vanished

x_o and m are function parameters (scale and shape parameters, respectively); and

y is the area or the volume of the specimen.

The integral is defined as the risk of fracture, R_{fr}.

The distribution of extreme values is influenced by the type of the function used to manipulate the data. Three types of asymptotic Extreme Value distributions have been developed.[2] The type I is unbounded in the direction of extreme value, the type II do not possess finite moments, and the type III is bounded in the direction of extreme value. The Weibull function use extreme value statistics and is a type III asymptotic extreme value distribution also known as Fisher–Tipper Type III distribution of smaller values and as the third asymptotic distribution of smallest extreme value.[3] Kittl and Diaz[4] proposed five mathematical ways for deduction of this function.

Another statistical function based in WLT theory where initiation of the failure takes place in the weakest part of the failure-prone volume is an extreme value function recently developed by Neville,[5–8] who tried to give a physical meaning to his function by considering the stress and strain near the crack tip. He considered that in a piece of material containing many microcracks, the failure-prone volume will increase in direct proportion to load. The failure-prone volume near the tip of the short length of crack front, δl, is linearly proportional to K^4, (K =stress intensity factor), or P^4 (P = load), and he expressed it as sampling S [$S = P^4 \times$(volume of piece), when all the pieces in a set have the same sizes $S = P^4$]. Neville considered a cumulative failure probability depending on the size of the surviving population and on some function $g(S)$. He supposed that his function has the following form:

$$g(S) = \left(\frac{S}{B_p}\right)^{D_p}$$

He developed the following expression as a simple statistical criterion for the failure of pieces containing microcracks, supposing that failure will take place when sampling is sufficiently reached. This sufficiency of sampling is statistically distributed according to the function:

$$P_f = \frac{\left(\frac{S}{B_p}\right)^{D_p}}{1 + \left(\frac{S}{B_p}\right)^{D_p}}$$

where: P_f is the cumulative probability of failure, S is the sampling, always linearly proportional to the actual amount of material loaded at greater than a given stress or strain. (It is defined as Ki^4, Ji^4, COD^2, and its effect, $g(S)$ is equal to $(S/B_p)^{Dp}$, B_p is a scale parameter which can change with temperature and which normalises S (it is actually the median value of S for all values of D_p), D_p is a shape parameter which allows the shape to change with temperature. Different D_p values for a given material are also expected for different distributions of stresses.

In order for the fracture probability to be expressed an estimator must be used. The expected values of the probability of failure P_f can be obtained from the two estimators used by Trustrum and Jayatilaka:[9]

$$P_f = \frac{i}{(N+1)}$$

and

$$P_f + \frac{(i - 0{\cdot}5)}{N}$$

where i is the ith (ranked) order of failure; and N is the total number of specimens experimentally tested.

They demonstrated that when less than 50 specimens are used for statistical analysis, the first of the above estimators give a more biased m. Other proposed estimators, (used also by Neville in his calculations[10]), follow the expressions:[11–13]

$$P_f = \frac{i - 0{\cdot}3}{(N + 0{\cdot}4)}$$

and

$$P_f = \frac{i - \frac{3}{8}}{\left(N + \frac{1}{4}\right)}$$

The first estimator corresponds to the median probability of failure. Examination of the statistical properties of the four estimators using Monte-Carlo simulation techniqu,[13] have shown that the popular $P_f = \frac{i}{(N+1)}$ estimator gives the more biased m. This means that the value of m is lower and gives a higher and so, more conservative probability of failure and from engineering point of view is the best choice in reliability prediction[13] and is used in many statistical applications.[14] The formula $P_f = (i + 0{\cdot}5)/N$ gives for N more than 20, the less biased m values and probably is the most preferable from a materials science point of view. Asloun *et al.*,[15] using the above estimator, have shown that the estimator and the sample size when it is higher than about 20 do not influence the results.

Results from flexural and tensile strength measurements on a carbon fibre–borosilicate composite[16] showed a poor fit of Weibull statistics, which was attributed to the small number (12) of tested specimens. Research has shown[11,17] that a sample size of about 30 specimens is acceptable for estimating Weibull parameters in ceramics and generally brittle materials.

2 Experimental Details

A series of brittle ceramic materials such as Remblend China clay from ECC International,[18] a pottery mixture and a brick clay were used for manufacturing all the test specimens. Details on fabrication, sintering programs, chemical, XRD analysis and AE analysis are given elsewhere.[19] All the batches of the 29, 63, 94 and 144 brick-clay samples were produced from the same raw material and under the same manufacturing and sintering conditions.

3 Results and Discussion

The effect of estimator type on Weibull Modulus was examined in several formulations and sintering temperatures and as an example the results on the K_{IC} values of kaolin notched specimens sintered at 1100°C and tested at 3-point bending, and are presented in Table 1. Graphical representation of the results are also shown in Fig. 1. The effect of sample size (population) was studied on 29, 63, 94 and 144 brick-clay samples fired at 900°C for 24 h and is presented in Table 2. The Weibull distribution curves for the different sample sizes are shown in Fig. 2.

Fracture data can be analyzed by applying probabilistic theories as well as deterministic theories

Table 1. Effect of type of Estimator on Weibull Modulus of K_{IC} for Kaolin Notched specimens sintered at 1100°C and tested at 3-point bending

Estimator	*Weibull modulus*
$\frac{i}{[N+1]}$	4·77
$\frac{i-0{\cdot}5}{N}$	5·20
$\frac{i-0{\cdot}3}{N+0{\cdot}4}$	5·01
$\frac{i-\frac{3}{8}}{N+\frac{1}{4}}$	5·08

i = ith (ranked) order of failure of specimen; N = total number of specimens.

like LEFM fracture theory. The observed wide scatter in ceramic strength, resulting in low reliability and high inconsistency of design strength data, is due mainly to inherent micro- and macro-structural chemical inhomogeneity and the presence of microcracks with various orientations, a large range of flaw sizes and the residual stresses arising from anisotropic contraction during cooling (SEM micrographs, Fig. 3). These characteristics makes the use of statistical analysis functions in order to get meaningful results vital.

Before any attempt at statistical analysis two parameters must be specified. The first one is what estimator should be used for estimation of the failure probability P_f and the second one is the minimum number of specimens which should be used for statistical analysis.

The first parameter which influences more or less analysis is the probability of failure P_f, a variable which goes straight into the statistical equation. The most appropriate estimator from those used most often was selected for examination of K_{IC} results from notched kaolin specimens sintered at 1100°C and fractured at 3-point bending (Table 1, Fig. 2). Graphical representation shows that the slope of the linearly transformed function decreases

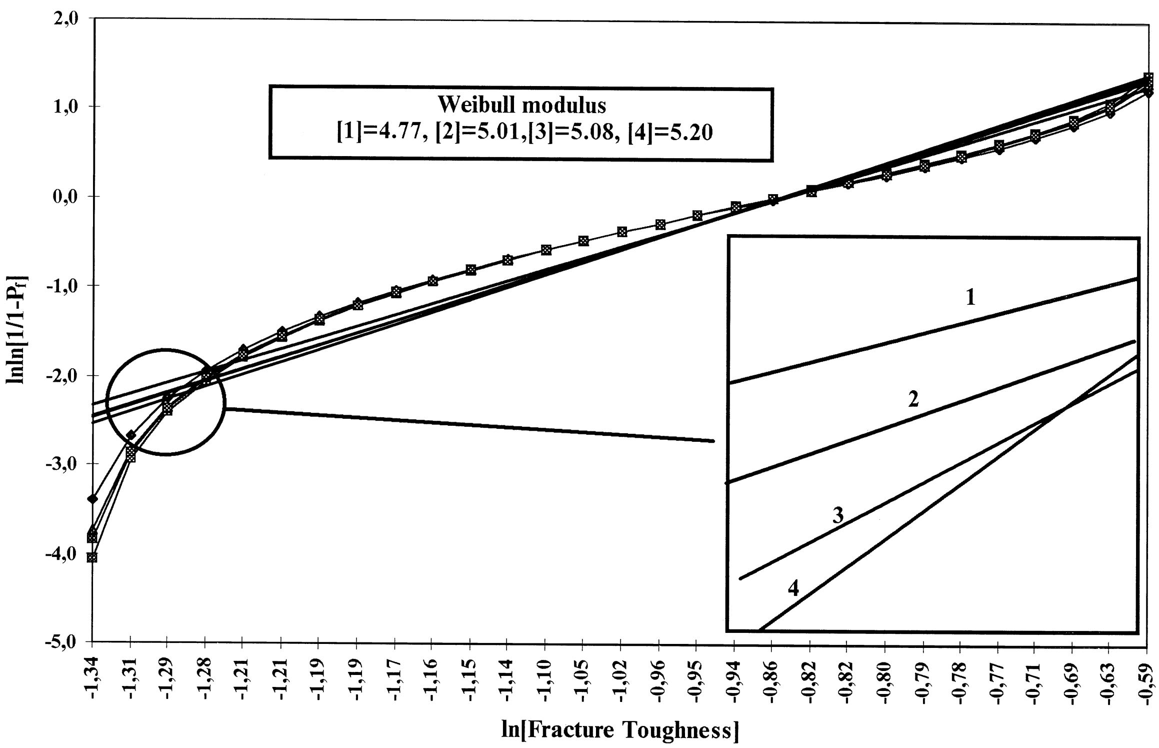


Fig. 1. Graphic representation of the effect of estimator type on Weibull modulus of notched Kaolin specimens sintered at 1100°C and tested at 3-point bending: 1. $i/(N+1)$, 2. $(i-0{\cdot}3)/(N+0{\cdot}4)$, 3. $(i-3/8)/((N+1/4)$, 4. $(i-0{\cdot}5)/N$. P_f = Probability of fracture, fracture toughness K_{IC} in MPa.m$^{1/2}$.

Table 2. Variation of statistical characteristics of brick-clay unnotched specimens sintered at 900°C as a function of the number of tested specimens.

Ns	*MOR*	*m*	r_o	r_I	r_N	MOR_{th} *(Mpa)*
29	8·54 [1·16]	8·54	0·9882	0·9868	0·9765	5·31
63	8·69 [1·01]	9·66	0·9931	0·9899	0·9719	5·58
94	9·16 [1·13]	9·35	0·9908	0·9934	0·9829	5·92
144	9·7 [1·34]	8·52	0·9848	0·9958	0·9844	5·52

Ns = number of tested specimens; m = Weibull modulus from 2-parameter Weibull function; r_o = correlation coefficient in a 2-parameter Weibull function; r_I = correlation coefficient in a 3-parameter Weibull function; r_N = Neville correlation coefficient; MOR_{th} = threshold value of MOR.

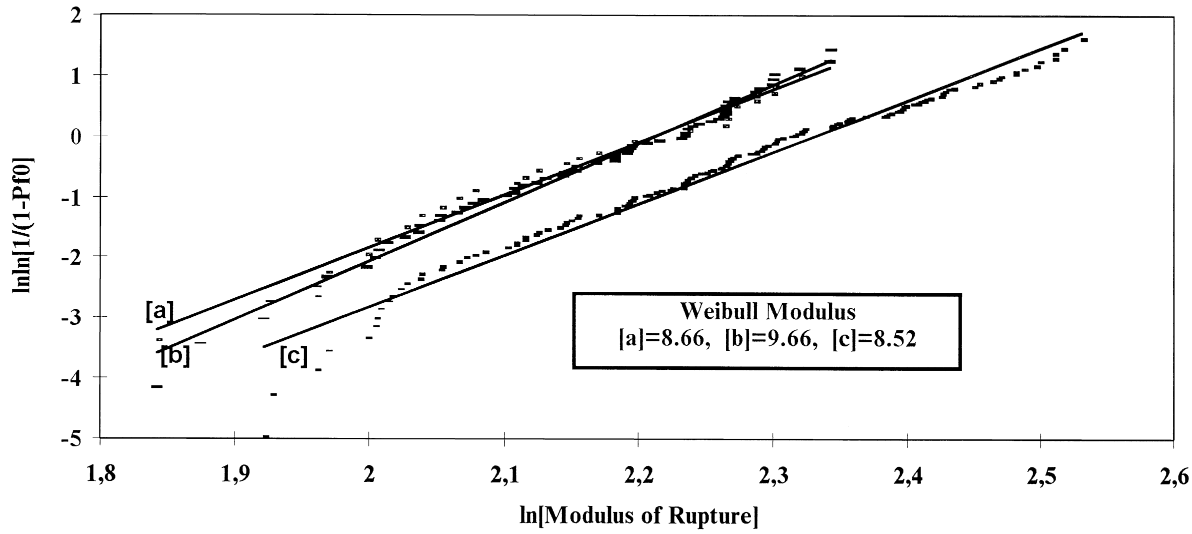
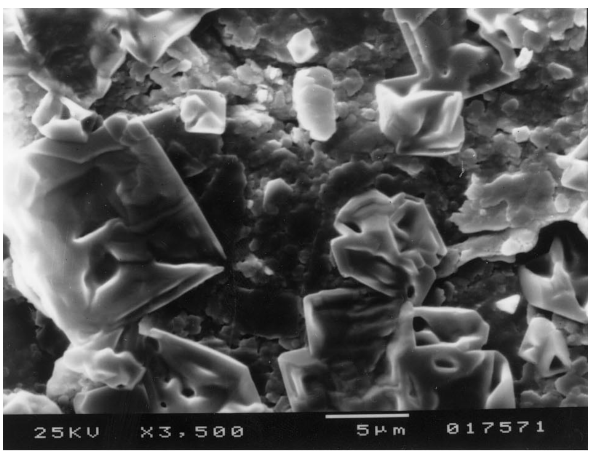
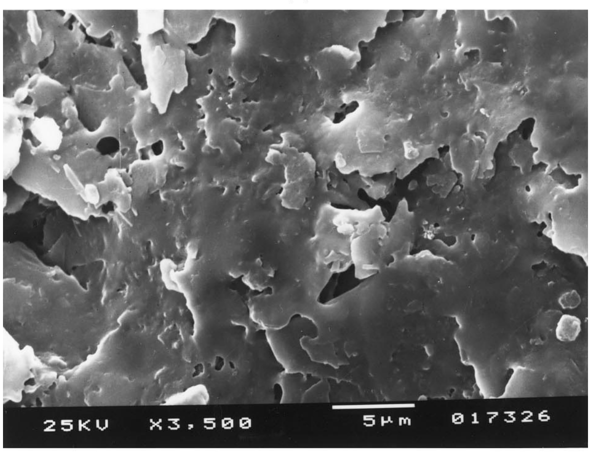


Fig. 2. Graphic representation of the effect of specimen population on Weibull modulus of brick-clays sintered at 900°C and tested at 3-point bending. Specimens: (a) 29, (b) 63, (c) 144. Modulus of rupture (MOR) in MPa.



(a)



(b)

Fig. 3. (a) SEM micrograph from fracture of kaolin specimen sintered at 1100°C and tested at 3-point bending; (b) SEM micrograph from fractured brick-clay specimen sintered at 900°C and tested at 3-point bending.

in the order: $(i{-}0{\cdot}5/n)$, $(i{-}3/8)/(n+1/4)$, $(i{-}0{\cdot}3)/(n+0{\cdot}4)$, $i/(n+1)$. However, no matter what the estimator type is, its effect is small on the shape of the curve.

These results coincide with the results of other workers (e.g. Refs 13 and 14) and show that the well known $P_f + \frac{i}{[N+1]}$ estimator gives the more biased m values and is good for engineering predictions, since it gives the more conservative value (the lowest m value).

The $P_f = \frac{i-0{\cdot}5}{N}$ estimator gives the least bias to the m Weibull modulus, and the $P_f = \frac{i-0{\cdot}3}{[n+0{\cdot}4]}$ estimator was very close to the average value of the four examined estimators.

The second parameter, concerning the effect of specimen population size on statistics, was estimated by analyzing MOR data from unnotched brick-clay samples sintered at 900°C for 24 h (Table 2, Fig. 2). It seems that the Weibull modulus remains nearly the same for a range of specimen numbers starting from 29 and going up to 144. The variation of the characteristic values of MOR and Weibull Modulus m is quite acceptable (about 0·5 STD of AVG values, and about 5% Coefficient of Variation). The number of about 30 specimens for statistics was considered suitable for this analysis, in agreement with the published literature (e.g. Refs 11 and 17).

The 2-parameter and 3-parameter Weibull functions and the Neville function were compared for best fitting of results by optical examination of the curves and by using correlation coefficients.[19] The z_W and z_N values were not calculated using Fisher's transformation since the relatively low number of specimens used for statistical analysis does not give a clue for a definite likelihood. Testing the effect of population size on correlation coefficients (Table 2) shows that increase of the number of samples does not influence much the r values. Complete examination of the effect of the two estimators ($i{-}0{\cdot}5/n$ and $i{-}0{\cdot}3/n+0{\cdot}4$) on correlation coefficients shows that the r values do not change their relative order.

4 Conclusions

In most of the cases, literature survey shows results from simulated data using the Monte Carlo technique. In the present work, ceramic fracture data show that the $P_f = \frac{i-0{\cdot}5}{N}$ estimator gives the least bias to the m Weibull modulus, and the $P_f = \frac{i-0{\cdot}3}{[n+0{\cdot}4]}$ estimator is very close to the average value of the four examined estimators. The $P_f = \frac{i}{(N+1)}$ estimator gives the more conservative m and reliability predictions using this modulus, give lower reliability values and it this sense Weibull function itself is an 'inaccurate' reliability prediction function.

Examination of the effect of specimen population on the shape parameter has shown that m remains almost constant. Simple estimators and a minimum number of about 30 specimens are needed for statistical analysis. The so-called Weibull modulus (the shape parameter of Weibull function), no matter what it expresses, since there is no real connection to features in the structure, seems to be relatively constant in a ceramic irrespectively of estimator type. Similar comments could be made also for shape parameter in the Neville function.

Generally should be expected that a three parameter Weibull function will fit more closely than a two parameter Weibull function, according to the idea that the more the parameters of a function the better the fitness. However correlation coefficients show that this is not always the case (Table 2).

It seems that modulus remains almost constant in the range of the specimen numbers tested and a minimum number of about 30 specimens is acceptable for reliability predictions at least for the examined type of ceramic.

The confidence is not improved by increasing the size of the specimens over 30, and more work is needed in order to test the universality of this conclusion and the dependence or not on the tested material.

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